Mathware & Soft Computing 12 (2005) 155-167

A Characterization for Residuated Implications on $\mathcal{J}[0,1]$. Application to the L-Fuzzy Concept Theory.

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Abstract

In this paper, a new characterization for the interval-valued residuated fuzzy implication operators is presented, with which it is possible to use them in a simple and efficient way, since the calculation of the values of an interval-valued implication applicated to two intervals is reduced to the study of a fuzzy implication applicated to the extremes of these intervals. This result is very important in order to extract knowledge from an L-fuzzy context with incomplete information. Finally, some examples of interval-valued residuated fuzzy implications built by this constructive method and some properties are shown.

Key words: Interval-valued fuzzy implication operator, Fuzzy implication operator, Fuzzy R-implication, L-Fuzzy concept theory.

1 Introduction. The interval-valued L-fuzzy context

The need to find a characterization for the residuated implications between intervals arises when we attempt to solve the problem of the extraction of knowledge from an L-fuzzy context with incomplete information.

Our proposal to solve this problem consists of transforming the L-fuzzy context into an interval-valued L-fuzzy context, in which the membership functions of the sets of objects and attributes as well as the relation defined among them take values in the set of closed intervals $\mathcal{J}[0,1]$.

In this way, we introduce in the values with which we work certain degree of ambiguity or vagueness. The greater it is the amplitude of the interval, greater will be the ambiguity. We will associate the lack of information to the total vagueness, therefore when a value is not known, we will suppose that its membership grade will be the interval $[0, 1]$.

The interval-valued L-fuzzy contexts have been previously studied by Burusco and Fuentes-González ([4, 5]) using several elements as the *multisets* or the *theory of expertons*. We are going to focus here the study of the interval-valued L-fuzzy contexts from a different point of view. We will consider the notion of L-fuzzy concept defined from implication operators ([3]) and we will extend it to the case of interval-valued contexts and implication operators between intervals.

Definition 1. Let $(L, \leq)$ be a complete lattice with a negation $'$ and a t-conorm S, and let $\mathcal{J}[L]$ be the set of all the closed intervals in L. Let X and Y be the object and attribute sets respectively and R an interval-valued L-fuzzy relation defined between X and Y.
We define an *interval-valued L-fuzzy context* as a tuple $(\mathcal{J}[L], X, Y, R)$.

We will denote by $\mathcal{J}[L]^X$ and $\mathcal{J}[L]^Y$ the classes of interval-valued L-fuzzy sets associated with X and Y respectively.

Definition 2. Let $\boldsymbol{I}$ be an interval-valued fuzzy implication defined on the lattice of intervals $(\mathcal{J}[L], \leq)$. Given $A \in \mathcal{J}[L]^X$ and $B \in \mathcal{J}[L]^Y$ two interval-valued L-fuzzy sets, and $R \in \mathcal{J}[L]^{X\times Y}$ an interval-valued L-fuzzy relation, we define the *derivated sets* of A and B, denoted by $A_1 \in \mathcal{J}[L]^Y$ and $B_2 \in \mathcal{J}[L]^X$ respectively, by the expressions:

$$A_1(y) = \underset{x\in X}{inf}\{\boldsymbol{I}(A(x), R(x, y))\}$$

$$B_2(x) = \underset{y\in Y}{inf}\{\boldsymbol{I}(B(y), R(x, y))\}$$

The operators denoted by the subscripts 1 and 2 are said to be *the derivation operators*.

Definition 3. We define the *constructor operators* $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as:

$$\boldsymbol{\varphi}\colon \mathcal{J}[L]^X \longrightarrow \mathcal{J}[L]^X \quad /\boldsymbol{\varphi}(A) = A_{12}$$

$$\boldsymbol{\psi}\colon \mathcal{J}[L]^Y \longrightarrow \mathcal{J}[L]^Y \quad /\boldsymbol{\psi}(B) = B_{21}$$

where A_{12} and B_{21} represent the interval-valued L-fuzzy sets $(A_1)_2$ and $(B_2)_1$ respectively.

Proposition 1. ([1]) If the implication $\boldsymbol{I}$ is a residuated implication, then the constructor operators $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are clousure operators.

We are going to use these operators to give the following definition:

Definition 4. If $M \in fix(\boldsymbol{\varphi})$, then the pair (M, M_1) is said to be an *interval-valued L-fuzzy concept* of the interval-valued L-fuzzy context $(\mathcal{J}[L], X, Y, R)$.

We have used the set $fix(\boldsymbol{\varphi})$ to define the interval-valued L-fuzzy concepts, but we could do it with the set of fixed points of the operator $\boldsymbol{\psi}$ ([2]).

We are interested in these interval-valued L-fuzzy concepts because they will give us a complete information of the L-fuzzy context.

The calculation of them will consist of the calculation of the fixed points of the operator $\boldsymbol{\varphi}$ (or $\boldsymbol{\psi}$) and, therefore, the results will depend on the chosen implication to define the derivation operators. Besides, when we use a residuated implication, the constructor operator is a clausure operator and for any $A \in \mathcal{J}[L]^X$, $\boldsymbol{\varphi}(A)$ is a fixed point of $\boldsymbol{\varphi}$.

As we have seen, the use of residuated implications will facilitate in a great measure the calculation of the interval-valued L-fuzzy concepts and hence it will be very important to have a characterization for the interval-valued residuated fuzzy implication operators that allows us to use them in a simple way.

2 Some fuzzy operators

In order to approach the study of the interval-valued R-implications, we should previously recall the concepts of fuzzy implication and fuzzy R-implication. We will also remember in this section the definitions of t-norm, t-conorm, negation and implication operator related to intervals.

2.1 Fuzzy implication operators

A *fuzzy implication* is a function of the form:

$$I\colon [0,1] \times [0,1] \longrightarrow [0,1]$$

which for any truth values a and b of the given propositions p and q respectively, defines the truth value $I(a,b)$ of the proposition $p \Rightarrow q$.

Any fuzzy implication must be an extension of the classical implication $p \Rightarrow q$ on the domain $\{0,1\}$, that is, the following values must be obtained:

$$\begin{cases} I(0,0) = I(0,1) = I(1,1) = 1 \\ I(1,0) = 0 \end{cases}$$

Generalizing the properties of the classical implications we obtain a set of axioms that may be verified by the fuzzy implications ([10, 12]).

i.1. $x \le z \Longrightarrow I(x,y) \ge I(z,y) \quad \forall y \in [0,1]$

i.2. $y \le t \Longrightarrow I(x,y) \le I(x,t) \quad \forall x \in [0,1]$

i.3. $I(0,y) = 1 \quad \forall y \in [0,1]$

i.4. $I(x,1) = 1 \quad \forall x \in [0,1]$

i.5. $I(1, x) = x \quad \forall x \in [0, 1]$

i.6. $I(x, x) = 1 \quad \forall x \in [0, 1]$

i.7. $I(x, I(y, z)) = I(y, I(x, z)) \quad \forall x, y, z \in [0, 1]$

i.8. $I(x, y) = 1 \Longleftrightarrow x \leq y$

i.9. $I(x, y) \geq y \quad \forall x, y \in [0, 1]$

i.10. $I(x, y) = I(n(y), n(x)) \quad \forall x, y \in [0, 1]$ with a strong negation n

i.11. *Continuity* of the function I in $[0, 1] \times [0, 1]$

Different classes of implications appear in the literature ([10, 12]), among them we will emphasize the R-implications (implications defined from a t-norm).

Definition 5. The *R-implication (or residuated implication) associated to the t-norm* T ([12, 10]) is the implication defined by:

$$I(a, b) = \sup\{x \in [0, 1]/T(a, x) \leq b\} \qquad \forall a, b \in [0, 1]$$

where T is a t-norm on $[0, 1]$.

It is not difficult to verify that all the residuated implications verify the properties i.1, i.2, i.3, i.4, i.5, i.6 and i.9.

2.2 Operations between intervals

Let $\mathcal{J}[0, 1]$ be the set of all the closed intervals in $[0, 1]$, that is

$$\mathcal{J}[0, 1] = \{[a, b]/a, b \in [0, 1]\}$$

We will consider the following ordering defined on $\mathcal{J}[0, 1]$:

$$[a, b] \leq [c, d] \quad \Longleftrightarrow \quad a \leq c \text{ and } b \leq d$$

We can find in the literature, among others, the following operators defined on $\mathcal{J}[0, 1]$:

2.2.1 t-norms on $\mathcal{J}[0, 1]$

Definition 6. ([9]) A *t-norm* defined on the set of closed intervals $\mathcal{J}[0, 1]$ is a mapping

$$\mathcal{T} : \mathcal{J}[0, 1] \times \mathcal{J}[0, 1] \longrightarrow \mathcal{J}[0, 1]$$

fulfilling the following properties:

t.1. *Boundary condition:*

$$\mathcal{T}\left([a,b],[1,1]\right)=[a,b] \quad \forall [a,b]\in\mathcal{J}[0,1]$$

t.2. *Commutatity:*

$$\mathcal{T}\left([a,b],[c,d]\right)=\mathcal{T}\left([c,d],[a,b]\right) \quad \forall [a,b],[c,d]\in\mathcal{J}[0,1]$$

t.3. *Monotonic increasing:*

$$[c,d]\leq[e,f]\Longrightarrow\mathcal{T}\left([a,b],[c,d]\right)\leq\mathcal{T}\left([a,b],[e,f]\right) \quad \forall [a,b]\in\mathcal{J}[0,1]$$

t.4. *Associativity:*

$$\mathcal{T}\left([a,b],\mathcal{T}\left([c,d],[e,f]\right)\right)=\mathcal{T}\left(\mathcal{T}\left([a,b],[c,d]\right),[e,f]\right),\forall [a,b],[c,d],[e,f]\in\mathcal{J}[0,1]$$

Definition 7. Let T be a t-norm defined on the interval $[0,1]$, the *associated t-norm* on $\mathcal{J}[0,1]$ is defined as follows ([11]):

$$\boldsymbol{T}\colon\mathcal{J}[0,1]\times\mathcal{J}[0,1]\longrightarrow\mathcal{J}[0,1]$$

where

$$\boldsymbol{T}\left([a,b],[c,d]\right)=[T(a,c),T(b,d)]$$

It is immediate to verify that the operation defined in this way fulfills the properties of the t-norms defined on $\mathcal{J}[0,1]$.

The t-norm thus obtained is an extension of the t-norm on $[0,1]$, since if we identify each element $a\in[0,1]$ with the interval $[a,a]$:

$$\boldsymbol{T}\left([a,a],[b,b]\right)=[T(a,b),T(a,b)] \quad \forall a,b\in[0,1]$$

2.2.2 Implications on $\mathcal{J}[0,1]$

Cornelis and Deschrijver ([7]) define the interval-valued fuzzy implication operator as an extension of fuzzy implication operator to the set of intervals $\mathcal{J}[0,1]$:

Definition 8. An *interval-valued fuzzy implication* is a function $\boldsymbol{I}$ defined on the set of intervals $\mathcal{J}[0,1]$:

$$\boldsymbol{I}\colon\mathcal{J}[0,1]\times\mathcal{J}[0,1]\longrightarrow\mathcal{J}[0,1]$$

fulfilling the following boundary conditions:

- $\boldsymbol{I}([0,0],[0,0])=[1,1]$
- $\boldsymbol{I}([0,0],[1,1])=[1,1]$
- $\boldsymbol{I}([1,1],[1,1])=[1,1]$

- $\boldsymbol{I}([1,1],[0,0]) = [0,0]$

Moreover, as it appears in literature ([6, 7, 11]), in general we will demand to any interval-valued fuzzy implication to fulfill the following properties:

I.0 It must be an extension of the fuzzy implication, that is:

$$\text{If } \boldsymbol{I}([a,a],[b,b]) = [x,y], \text{ then } x = y$$

I.1 If $[a,b] \leq [a_1,b_1]$ then $\boldsymbol{I}([a,b],[c,d]) \geq \boldsymbol{I}([a_1,b_1],[c,d]) \quad \forall [c,d] \in \mathcal{J}[0,1]$

I.2 If $[c,d] \leq [c_1,d_1]$ then $\boldsymbol{I}([a,b],[c,d]) \leq \boldsymbol{I}([a,b],[c_1,d_1]) \quad \forall [a,b] \in \mathcal{J}[0,1]$

The interval-valued fuzzy implications can also verify a set of properties that some authors ([6, 11]) divide in two groups:

- Extension of the properties of fuzzy implications to the interval-valued case:

I.3 $\boldsymbol{I}([0,0],[a,b]) = [1,1] \quad \forall [a,b] \in \mathcal{J}[0,1]$

I.4 $\boldsymbol{I}([a,b],[1,1]) = [1,1] \quad \forall [a,b] \in \mathcal{J}[0,1]$

I.5 $\boldsymbol{I}([1,1],[a,b]) = [a,b] \quad \forall [a,b] \in \mathcal{J}[0,1]$

I.6 $\boldsymbol{I}([a,b],[a,b]) = [1,1] \quad \forall [a,b] \in \mathcal{J}[0,1]$

I.7 $\boldsymbol{I}([a,b],\boldsymbol{I}([c,d],[e,f])) = \boldsymbol{I}([c,d],\boldsymbol{I}([a,b],[e,f]))$
$\forall [a,b],[c,d],[e,f] \in \mathcal{J}[0,1]$

I.8 $\boldsymbol{I}([a,b],[c,d]) = [1,1]$ if and only if $[a,b] \leq [c,d] \quad \forall [a,b],[c,d] \in \mathcal{J}[0,1]$

I.9 $\boldsymbol{I}([a,b],[c,d]) \geq [c,d] \quad \forall [a,b],[c,d] \in \mathcal{J}[0,1]$

I.10 $\boldsymbol{I}([a,b],[c,d]) = \boldsymbol{I}([n(d),n(c)],[n(b),n(a)]) \quad \forall [a,b],[c,d] \in \mathcal{J}[0,1]$, where n is a strong negation.

- Special properties of the interval-valued fuzzy sets:

Let us denote by $W([a,b])$ the amplitude of the interval $[a,b]$, that is,
$W([a,b]) = b - a$.

I.11 $W(\boldsymbol{I}([a,b],[c,d])) \leq \bigvee(n(a),n(b),n(c),n(d)) = \bigvee(n(a),n(c))$ with n a strong negation.

I.12 If $[a,b] = [c,d]$ then $W(\boldsymbol{I}([a,b],[c,d])) = W([a,b])$

I.13 If $W[a,b] = W[c,d]$ then $W(\boldsymbol{I}([a,b],[c,d])) = W([a,b])$

From this point, we will focus on the study of the residuated implications.

3 Residuated Implications on $\mathcal{J}[0,1]$

Definition 9. Let T be a given t-norm on $[0,1]$, the *R-implication* associated to $\boldsymbol{T}$ on $\mathcal{J}[0,1]$ is defined by:

$$\boldsymbol{I_{R_T}}([a,b],[c,d]) = \sup\{[x,y]/\boldsymbol{T}([a,b],[x,y]) \leq [c,d]\}$$

where $\boldsymbol{T}$ represents the extension of the t-norm T to $\mathcal{J}[0,1]$ defined in the section 2.2.1.

In the following we will show that every interval-valued residuated fuzzy implication can be characterized from a fuzzy R-implication.

Theorem 1. Every residuated implication between intervals $\boldsymbol{I_{R_T}}$ associated to the t-norm $\boldsymbol{T}$ has the form:

$$\boldsymbol{I_{R_T}}([a,b],[c,d]) = [I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] \tag{1}$$

where I_{R_T} is the residuated implication associated to the t-norm T on $[0,1]$ which generates $\boldsymbol{T}$ on $\mathcal{J}[0,1]$.

Proof.
In general, every residuated implication between intervals has the form:

$$\begin{aligned} \boldsymbol{I_{R_T}}([a,b],[c,d]) &= \sup\{[x,y]/ \quad \boldsymbol{T}([a,b],[x,y]) \leq [c,d]\} \\ &= \sup\{[x,y]/ \quad T(a,x) \leq c \quad \& \quad T(b,y) \leq d\} \end{aligned} \tag{2}$$

We are going to prove that (1) and (2) expressions are equal.

Let M be the set:

$$M = \{[x,y]/ \quad T(a,x) \leq c \quad \& \quad T(b,y) \leq d\}$$

If we recall the definition of residuated implication on $[0,1]$

$$I_{R_T}(a,c) = \sup\{x/ \quad T(a,x) \leq c\}$$

then, we obtain

$$\forall [x,y] \in M \quad T(a,x) \leq c \quad \& \quad T(b,y) \leq d \Longrightarrow$$

$$\Longrightarrow I_{R_T}(a,c) \geq x \quad \& \quad I_{R_T}(b,d) \geq y \geq x$$

Therefore, if we consider the minimum, then it follows that:

$$[I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] \geq [x,y] \quad \forall [x,y] \in M$$

and hence, we have proved that the second member of the equality (1) is an upper bound of M:

$$[I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] \geq \boldsymbol{I_{R_T}}([a,b],[c,d]) \tag{3}$$

Let us see that it is the least upper bound of M:

Suppose that it is not. Let $[\alpha,\beta]$ be another upper bound of M which is not greater than the last one, then:

$$[\alpha,\beta] \geq [x,y] \quad \forall [x,y] \in M \tag{4}$$

$$[I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] \not\leq [\alpha,\beta]$$

it is deduced then one of these three cases:

a)

$$\left.\begin{array}{rcl} \alpha & < & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \\ \beta & < & I_{R_T}(b,d) \end{array}\right\}$$

b)

$$\left.\begin{array}{rcl} \alpha & < & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \\ \beta & \geq & I_{R_T}(b,d) \end{array}\right\}$$

c)

$$\left.\begin{array}{rcl} \alpha & \geq & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \\ \beta & < & I_{R_T}(b,d) \end{array}\right\}$$

Let us consider the cases:

a) If $\left.\begin{array}{rcl} \alpha & < & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \\ \beta & < & I_{R_T}(b,d) \end{array}\right\}$ then

$$\left.\begin{array}{rcl} \alpha & < & I_{R_T}(a,c) \\ \alpha & < & I_{R_T}(b,d) \\ \beta & < & I_{R_T}(b,d) \end{array}\right\}$$

applying the definition of residuated implication

$$\left.\begin{array}{l} \alpha < \sup\{x/T(a,x) \leq c\} \\ \alpha < \sup\{x/T(b,x) \leq d\} \\ \beta < \sup\{y/T(b,y) \leq d\} \end{array}\right\}$$

and then

$$\left.\begin{array}{l} \exists z_1 > \alpha/T(a,z_1) \leq c \\ \exists z_2 > \alpha/T(b,z_2) \leq d \\ \exists z_3 > \beta \geq \alpha/T(b,z_3) \leq d \end{array}\right\}$$

Let $r = z_1 \wedge z_2$ and $s = z_2 \vee z_3$, it is verified that

$$r \leq z_2 \leq s \Rightarrow r \leq s$$

we have that

- If $z_2 < z_3$ then $s = z_3$ and $T(b,s) = T(b,z_3) \leq d$
- If $z_2 \geq z_3$ then $s = z_2$ and in this case $T(b,s) = T(b,z_2) \leq d$

and, on the other hand:

- If $z_1 < z_2$ then $r = z_1$ and $T(a,r) = T(a,z_1) \leq c$
- If $z_1 \geq z_2$ then $r = z_2$ and by the monotonicity of the t-norm, $T(a,r) = T(a,z_2) \leq T(a,z_1) \leq c$

Therefore, $[r,s] \in M$ and it verifies $[r,s] \not\leq [\alpha,\beta]$, which contradicts (4)

b) If $\left.\begin{array}{lll} \alpha & < & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \\ \beta & \geq & I_{R_T}(b,d) \end{array}\right\}$

We have, hence, that:

$$\alpha < I_{R_T}(a,c) \wedge I_{R_T}(b,d) \Longrightarrow \left.\begin{array}{lll} \alpha & < & I_{R_T}(a,c) \\ \alpha & < & I_{R_T}(b,d) \end{array}\right\}$$

and by the definition of R-implication

$$\left.\begin{array}{l} \alpha < \sup\{x/T(a,x) \leq c\} \\ \alpha < \sup\{x/T(b,x) \leq d\} \end{array}\right\}$$

Therefore

$$\left.\begin{array}{l} \exists z_1 > \alpha/T(a,z_1) \leq c \\ \exists z_2 > \alpha/T(b,z_2) \leq d \end{array}\right\}$$

Let $r = z_1 \wedge z_2$. It is verified:

- If $z_1 < z_2$ then $r = z_1$ and therefore $\left\{\begin{array}{l} T(a,r) = T(a,z_1) \leq c \\ T(b,r) = T(b,z_1) \leq T(b,z_2) \leq d \end{array}\right.$
- If $z_1 \geq z_2$ then $r = z_2$, therefore $\left\{\begin{array}{l} T(b,r) = T(b,z_2) \leq d \\ T(a,r) = T(a,z_2) \leq T(a,z_1) \leq c \end{array}\right.$

Consequently, $[r,r] \in M$ and it verifies $[r,r] \not\leq [\alpha,\beta]$, which contradicts (4).

c) If $\left.\begin{array}{lll} \alpha & \geq & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \\ \beta & < & I_{R_T}(b,d) \end{array}\right\}$ is verified.

First, we have that $\alpha \leq \beta$ and $\beta < I_{R_T}(b,d)$, then $\alpha < I_{R_T}(b,d)$.

Therefore,

$$\left.\begin{array}{lcl} \alpha & < & I_{R_T}(b,d) \\ \alpha & \geq & I_{R_T}(a,c) \wedge I_{R_T}(b,d) \end{array}\right\} \Longrightarrow \alpha \geq I_{R_T}(a,c)$$

and since I_{R_T} is a residuated implication,

$$\alpha \geq x \quad \forall x / T(a,x) \leq c$$

Let us fix any value r such that $T(a,r) \leq c$, then $\alpha \geq r$.
Moreover,

$$\beta < I_{R_T}(b,d) \Longrightarrow \beta < \sup\{y/T(b,y) \leq d\} \Longrightarrow$$
$$\exists z > \beta \geq \alpha / T(b,z) \leq d$$

We obtain then: $r \leq \alpha \leq \beta < z$ and, furthermore, $\left\{\begin{array}{l} T(a,r) \leq c \\ T(b,z) \leq d \end{array}\right.$
Hence, $[r,z] \in M$ and it verifies $[r,z] \not\leq [\alpha,\beta]$, which contradicts (4).

We have proved that

$$[I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] = \sup\{[x,y]/ \quad T([a,b],[x,y]) \leq [c,d]\}$$

that is:

$$[I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] = \boldsymbol{I_{R_T}}([a,b],[c,d]) \quad \square$$

We will illustrate the previous theorem with some notable examples.

Examples.

1. If the chosen t-norm is the minimum $T(x,y) = \min(x,y)$, then we obtain the implication $\boldsymbol{I_{R_B}}$, which is the extension of the *Brouwer-Gödel* implication:

$$\boldsymbol{I_{R_B}}([a,b],[c,d]) = [I_{R_B}(a,c) \wedge I_{R_B}(b,d), I_{R_B}(b,d)]$$

where:

$$I_{R_B}(a,b) = \sup\{x/\min(a,x) \leq b\} = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

and then it is obtained that:

$$\boldsymbol{I_{R_B}}([a,b],[c,d]) = \begin{cases} [c,d] & \text{if } a > c \text{ and } b > d \\ [c,1] & \text{if } a > c \text{ and } b \leq d \\ [1,1] & \text{if } a \leq c \text{ and } b \leq d \\ [d,d] & \text{if } a \leq c \text{ and } b > d \end{cases}$$

2. If we use the t-norm $T(x, y) = \max(0, x + y - 1)$, then we obtain the implication $\boldsymbol{I_{R_L}}$, which is the extension of the *Lukasiewicz* implication:

$$\boldsymbol{I_{R_L}}([a,b],[c,d]) = [I_{R_L}(a,c) \wedge I_{R_L}(b,d), I_{R_L}(b,d)]$$

where:

$$I_{R_L}(a,b) = \min(1, 1 - a + b)$$

and therefore,

$$\boldsymbol{I_{R_L}}([a,b],[c,d]) = [\min(1, 1-a+b, 1-b+d), \min(1, 1-b+d)] =$$
$$= \begin{cases} [(1-a+c) \wedge (1-b+d), (1-b+d)] & \text{if } a > c \text{ and } b > d \\ [1-a+c, 1] & \text{if } a > c \text{ and } b \le d \\ [1,1] & \text{if } a \le c \text{ and } b \le d \\ [1-b+d, 1-b+d] & \text{if } a \le c \text{ and } b > d \end{cases}$$

We can find in [8] the intuitionistic version of these implications.

The R-implication $\boldsymbol{I_{R_T}}$ verifies the properties of the interval-valued fuzzy implications which are extension of the properties verified by the R-implication I_{R_T}, that is, it verifies the properties: I.1, I.2, I.3, I.4, I.5, I.6 and I.9, as we are going to show in the following

Proposition 2.

I.1 If $[a,b] \le [a_1,b_1]$ then $\boldsymbol{I_{R_T}}([a,b],[c,d]) \ge \boldsymbol{I_{R_T}}([a_1,b_1],[c,d]) \quad \forall [c,d] \in \mathcal{J}[0,1]$.

I.2 If $[c,d] \le [c_1,d_1]$ then $\boldsymbol{I_{R_T}}([a,b],[c,d]) \le \boldsymbol{I_{R_T}}([a,b],[c_1,d_1]) \quad \forall [a,b] \in \mathcal{J}[0,1]$.

I.3 $\boldsymbol{I_{R_T}}([0,0],[a,b]) = [1,1] \quad \forall [a,b] \in \mathcal{J}[0,1]$.

I.4 $\boldsymbol{I_{R_T}}([a,b],[1,1]) = [1,1] \quad \forall [a,b] \in \mathcal{J}[0,1]$.

I.5 $\boldsymbol{I_{R_T}}([1,1],[a,b]) = [a,b] \quad \forall [a,b] \in \mathcal{J}[0,1]$.

I.6 $\boldsymbol{I_{R_T}}([a,b],[a,b]) = [1,1] \quad \forall [a,b] \in \mathcal{J}[0,1]$.

I.9 $\boldsymbol{I}([a,b],[c,d]) \ge [c,d] \quad \forall [a,b],[c,d] \in \mathcal{J}[0,1]$

Proof.

I.1 $[a,b] \le [a_1,b_1] \iff \begin{cases} a \le a_1 \\ b \le b_1 \end{cases}$

And by the monotonicity of the implication I_{R_T} we have

$$\begin{cases} I_{R_T}(a,c) \ge I_{R_T}(a_1,c) \\ I_{R_T}(b,d) \ge I_{R_T}(b_1,d) \end{cases} \Longrightarrow$$
$$[I_{R_T}(a,c) \wedge I_{R_T}(b,d), I_{R_T}(b,d)] \ge [I_{R_T}(a_1,c) \wedge I_{R_T}(b_1,d), I_{R_T}(b_1,d)] \Longrightarrow$$
$$\Longrightarrow \boldsymbol{I_{R_T}}([a,b],[c,d]) \ge \boldsymbol{I_{R_T}}([a_1,b_1],[c,d])$$

I.2 The proof is similar to the one above applying that the implication I_{R_T} is increasing in its second argument.

I.3 The proof is obvious keeping in mind that $I_{R_T}(0, y) = 1 \quad \forall y \in [0, 1]$.

I.4 Evident since $I_{R_T}(x, 1) = 1 \quad \forall x \in [0, 1]$.

I.5 If we apply that it is verified that $I_{R_T}(1, y) = y \quad \forall y \in [0, 1]$, we obtain

$$\boldsymbol{I_{R_T}}([1, 1], [a, b]) = [I_{R_T}(1, a) \wedge I_{R_T}(1, b), I_{R_T}(1, b)] = [a \wedge b, b] = [a, b]$$

I.6 The proof is evident keeping in mind that $I_{R_T}(x, x) = 1 \quad \forall x \in [0, 1]$

I.9 Applying that the implication I_{R_T} fulfills that $I_{R_T}(x, y) \geq y \quad \forall x, y \in [0, 1]$, we obtain:

$$\boldsymbol{I_{R_T}}([a, b], [c, d]) = [I_{R_T}(a, c) \wedge I_{R_T}(b, d), I_{R_T}(b, d)] \geq [c \wedge d, d] = [c, d]. \qquad \square$$

4 Conclusions

This paper is a part of a wider work about the study of the information derived from the interval-valued L-fuzzy contexts. These contexts are represented by Fuzzy relations, which values are intervals, from wich we extract relevant information through the L-fuzzy concept theory developed by us, and using implications between intervals. To do it, we have to use some operations between intervals as these defined here.

Specifically, to calculate an L-Fuzzy concept is necessary the reiterated use of an interval-valued L-Fuzzy implicacion, and this calculation is simplified to a great extent when the used implication is a residuated implication.

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